

# Notes on the Schwinger model: regularization and gauge invariance

## Abstract

The point-splitting computation of the gauge invariant Hamiltonian for the Schwinger model on the circle in a positive energy representation is presented.

## 1 Notation

The starting point is the action functional

$$S = \int \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\hbar c \not{D}_A - \frac{mc^2}{\hbar}) \psi \right] dx dt \quad (1.1)$$

describing the interaction of a Dirac spinor field  $\psi$  with an electromagnetic field in Minkowski space-time with coordinates  $(x^0 = ct, \mathbf{x})$  and metric  $c^2 dt^2 - d\mathbf{x}^2$ . The electromagnetic field is given in terms of the potential  $A$  1-form by  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . The Dirac operator is given by

$$\not{D}_A = \gamma^\mu (\partial_\mu - \frac{ie}{\hbar c} A_\mu) \quad (1.2)$$

where we use the gamma matrices in the form (1.5). As in e.g. [5] we write  $\bar{\psi} = \psi^\dagger \gamma^0$ , where  $\dagger$  means Hermitian conjugate. The equations of motion are

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -e \bar{\psi} \gamma^\nu \psi, \\ i \not{D}_A \psi &= \frac{mc}{\hbar} \psi. \end{aligned} \quad (1.3)$$

We work in  $1+1$  dimensional space-time with coordinates  $(x^0 = ct, x^1 = x)$  and metric  $(dx^0)^2 - dx^2 = c^2 dt^2 - dx^2$ , with  $0 \leq x \leq L$  and periodic boundary conditions. Using  $x^0$  as the time variable and rescaling  $A, \psi, x$  gives the equations in natural units (with  $\hbar = 1 = c$ ). Further, writing  $A = ea$  removes the coupling constant from the interaction so that it appears only in the denominator in front of the first term in the action (and in the commutation relation). As in [3] we will work in the radiation gauge of Fermi, in which the spatial component of the connection  $a$  depends only on time so that the expression for the electric field  $E = \dot{a} - \partial a_0$  is in fact the decomposition into the longitudinal and transverse components  $E^{long} = -\partial a_0$  and  $E^{tr} = \dot{a}$  respectively. The time component  $a_0$  is integrated out via the Gauss law leading to the following classical Hamiltonian in the zero mass case:

$$H = \int_0^L \frac{1}{2e^2} \dot{a}^2 - \psi^\dagger (i\gamma^5 (\partial - ia) \psi) + \frac{1}{2} e^2 (\psi^\dagger \psi) (-\Delta)^{-1} * (\psi^\dagger \psi) dx. \quad (1.4)$$

Here  $(-\Delta)^{-1}$  means the kernel of the operator  $-\Delta = -\partial^2$  on  $[0, L]$  with periodic boundary conditions,  $*$  is convolution and  $\partial = \partial_x$ . Notice that  $E^{long} = -\partial a_0$ , the longitudinal component of the electric field, has been integrated out leaving only the transverse component  $E^{tr} = \dot{a}$ . We use the following form of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (1.5)$$

and we will use dots (resp.  $\partial$  or primes) to indicate derivatives with respect to  $t$  (resp.  $x$ ).

In this formulation there is a residual gauge invariance by “large” gauge transformations  $g_N(x) = e^{2\pi i N x/L}$  for  $N \in \mathbb{Z}$ , which play an important role.

To quantize the theory it is necessary to associate operators to the fields which satisfy the canonical relations:

$$\{\psi_\alpha(t, x), \psi_\beta^\dagger(t, y)\} = \delta_{\alpha\beta}\delta(x - y) \quad (1.6)$$

(other anti-commutators being zero) and

$$[E^{tr}, a] = [\dot{a}, a] = -\frac{ie^2}{L} \quad (1.7)$$

(other commutators being zero). In the process of quantizing it is necessary to define carefully what is meant the various bilinear quantities such as the axial charge and the Hamiltonian itself. The end point will be the following formula for the Hamiltonian:

$$\begin{aligned} H = & -\frac{e^2}{L} \frac{d^2}{da^2} + \sum |k_m| (b_m^\dagger b_m + c_m^\dagger c_m) \\ & a \left( \sum_{m \geq 0} b_m^\dagger b_m + \sum_{m < 0} b_m^\dagger b_m - \sum_{m > 0} c_m^\dagger c_m - \sum_{m \leq 0} c_m^\dagger c_m \right) \\ & + \frac{a^2 L}{2\pi} + a + \frac{e^2 L}{2} \sum_{m \neq 0} \frac{1}{k_m^2} j_0(-m) j_0(+m). \end{aligned} \quad (1.8)$$

Here  $b_m^\dagger, b_m$  (resp.  $c_m^\dagger, c_m$ ) are fermionic (resp. anti-fermionic) creation, annihilation operators acting on the zero charge fermionic Fock space  $\mathcal{H}_0$ , with non-interacting vacuum  $\Omega_0$ , and  $a$  is represented as the operator of co-ordinate multiplication (Schrödinger representation) in the Hilbert space

$$\mathcal{K} = \{\Psi = \Psi(a) \in \mathcal{H}_0 : \Psi \in L^2([0, \frac{2\pi}{L}], da; \mathcal{H}_0)\}. \quad (1.9)$$

Finally the operators  $j_0(m)$  are Fourier modes of the current operator associated to the classical current  $j_0 = \psi^\dagger \psi$ , see (3.37)-(3.39). The last term in (1.8) is just the operator associated to the Coulomb energy, (which will also require its vacuum expectation to be subtracted.)

## 2 Solution of the classical Schwinger model

The classical equations of motion are

$$\begin{aligned} i\dot{\psi} &= -i\gamma^5(\partial\psi - ia\psi) - a_0\psi \\ -\Delta a_0 &= -e^2 \psi^\dagger \psi = -e^2 j^0, \\ -\dot{E} &= -e^2 \psi^\dagger \gamma^5 \psi = -e^2 j^1, \end{aligned} \quad (2.1)$$

where the electric field is  $E = F_{01} = \dot{a} - \partial a_0$ . These can be reduced to linear equations as follows: let  $\varphi$  be a solution of the free Dirac equation

$$i\dot{\varphi} = -i\gamma^5 \partial \varphi \quad (2.2)$$

and write  $\psi = e^{i(f+g\gamma^5)}\varphi$  with  $f, g$  two real-valued functions, then the first equation of (2.1) is equivalent to the pair of equations

$$\begin{aligned} -\dot{f} &= \partial g - a_0 \\ -\dot{g} &= -a + \partial f. \end{aligned} \quad (2.3)$$

Given  $(a_0, a)$  solutions of these may be generated by first solving the inhomogeneous wave equation  $\ddot{f} - \partial^2 f = \dot{a}_0$  and then defining  $g = \int_0^t (a(s, x) + \partial f(s, x)) ds$ . To complete the reduction to linear equations we now observe that the currents  $j^0$  and  $j^1$  defined in the second two equations of (2.1) obey the charge and axial charge conservation laws:

$$\partial_\mu j^\mu = 0 \quad \text{and} \quad \partial_\mu j^{5\mu} = 0 \quad (2.4)$$

where  $j = (j^0, j^1)$  and  $j^5 = (j^1, j^0)$ . (These are the conservation laws arising, respectively, from phase invariance - under the transformation  $\psi \rightarrow e^{i\theta}\psi$ , and axial phase invariance - under the transformation  $\psi \rightarrow e^{i\gamma^5\theta}\psi$ .) (2.4) implies that both  $j^0$  and  $j^1$  solve the homogeneous wave equation

$$\square j^\mu = 0,$$

and  $a_0, a$  are then determined by solving the second two equations of (2.1). In fact  $j^0, j^1$  can be written in terms of the free Dirac field  $\varphi$  as  $j^0 = \varphi^\dagger \varphi$  and  $j^1 = \varphi^\dagger \gamma^5 \varphi$  as a consequence of  $(\gamma^5)^\dagger = -\gamma^1 \gamma^0 = +\gamma^0 \gamma^1 = \gamma^5$ . We write  $Q = \int j^0 dx$  and  $Q^5 = \int j^1 dx$  for the corresponding conserved charges - the first of these is electric charge and the second will be referred to as axial charge.

### 3 Quantization of the Dirac field in an external potential

Using the infinite Dirac sea representation, second quantization would associate to the field  $\psi$  an operator

$$\psi = \sum \left( a_n^R u^R e^{ik_n x} + a_n^L u^L e^{ik_n x} \right)$$

where  $\{a_n^R, a_{n'}^{R,\dagger}\} = \{a_n^L, a_{n'}^{L,\dagger}\} = \delta_{nn'}$ , all other anti-commutators being zero; formally this ensures that (1.6) holds. The  $u^{L,R}$  are eigenvectors of  $\gamma^5$  with  $\gamma^5 u^R = u^R$  and  $\gamma^5 u^L = -u^L$ . Using this representation the free Dirac Hamiltonian is

$$\sum k_n (a_n^{R,\dagger} a_n^R - a_n^{L,\dagger} a_n^L) \quad (3.1)$$

which fails to be non-negative, or even bounded below. The physical states are supposed to be those in which all but a finite number of negative energy states are occupied. In particular the states in which all right-moving states with  $n \leq P$  and all left-moving states with  $n \geq P+1$  filled are referred to as *unexcited* states in [3]. In this representation it is necessary to regularize operators such as the charge and Hamiltonian in order to get finite expectation values on such states; gauge invariant heat kernel regularization or some variant is often used. Introducing the positive energy representation in which fermions and anti-fermions are both explicitly present, it is possible to avoid using any extreme cut-off such as heat kernel regularization and instead use only Schwinger gauge invariant point-splitting to define the bilinear quantities. Thus we define

$$\begin{aligned} b_n &= a_n^R & (n \geq 0) \\ b_n &= a_n^L & (n < 0) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} c_n &= a_{-n}^{R,\dagger} & (n > 0) \\ c_n &= a_{-n}^{L,\dagger} & (n \leq 0) \end{aligned} \quad (3.3)$$

so that an unoccupied right (left) state of negative (positive) wave number is now re-interpreted as a state filled by a right (left) moving anti-fermion. The relations (1.6) can be guaranteed by writing

$$\psi = \frac{1}{\sqrt{L}} \sum_{n \in \mathbb{Z}} (b_n u_n e^{ik_n x} + c_n^\dagger v_n e^{-ik_n x}), \quad k_n = \frac{2n\pi}{L} \quad (3.4)$$

with

$$\{b_n, b_{n'}^\dagger\} = \{c_n, c_{n'}^\dagger\} = \delta_{nn'} \quad (3.5)$$

(other anti-commutators being zero) and

$$\begin{aligned} u_n &= u^R \mathbb{1}_{\{n \geq 0\}} + u^L \mathbb{1}_{\{n < 0\}}, \\ v_n &= u^R \mathbb{1}_{\{n > 0\}} + u^L \mathbb{1}_{\{n \leq 0\}}. \end{aligned} \quad (3.6)$$

There is a (non-interacting) vacuum  $\Omega_0$  and associated states

$$\Omega_{\mathbf{m},\mathbf{n}} = \Pi b_{m_i}^\dagger c_{n_j}^\dagger \Omega_0 \quad (3.7)$$

where  $\mathbf{m} = \{m_i\}_{i=1}^M$  and  $\mathbf{n} = \{n_j\}_{j=1}^N$  range over finite subsets of  $\mathbb{Z}$ . Let  $\mathcal{F}$  be the linear span of all the  $\Omega_{\mathbf{m},\mathbf{n}}$ , let  $\mathcal{F}_0 \subset \mathcal{F}$  be the zero charge subspace in which there are equal numbers of fermions and anti-fermions, and finally let  $\mathcal{F}_0^P \subset \mathcal{F}_0$  be the subspace in which  $Q^5 = P$  so that  $\mathcal{F}_0 = \cup_{P \in \mathbb{Z}} \mathcal{F}_0^P$ , where  $Q^5$  is as in (3.10).

In this representation the *unexcited* states arise when  $M = N = P \in \mathbb{Z}^+$  as follows. The case  $P = 0$  corresponds to the vacuum  $\Omega_0$ . For  $P \geq 0$  let  $m_i = n_i = i$  for  $0 \leq i \leq P$ , and define the unexcited state

$$\Omega_{P+1} = \prod_{i=0}^P \prod_{j=0}^P b_{m_i}^\dagger c_{n_j}^\dagger \Omega_0 \quad (P \in \mathbb{Z}^+); \quad (3.8)$$

for  $P < 0$  and  $M = N = -P$  let  $m_i = n_i = -i$  for  $0 < i \leq -P$ , and define the unexcited state

$$\Omega_P = \prod_{i=-1}^P \prod_{j=-1}^P b_{m_i}^\dagger c_{n_j}^\dagger \Omega_0 \quad (P \in \mathbb{Z}^- - \{0\}). \quad (3.9)$$

For the free (i.e. in the absence of the potential  $a_0, a$ ) Dirac field, in this representation, the charge and axial charge are given by

$$\begin{aligned} Q &= \sum b_n^\dagger b_n - c_n^\dagger c_n, \\ Q^5 &= \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n, \end{aligned} \quad (3.10)$$

and the Hamiltonian is

$$H_D^0 = \sum |k_n| (b_n^\dagger b_n + c_n^\dagger c_n) \quad (3.11)$$

of which the  $\Omega_{\mathbf{m},\mathbf{n}}$  are eigenvectors with eigenvalues  $(\sum_{i=1}^M |k_{m_i}| + \sum_{j=1}^N |k_{n_j}|)$ . Thus the  $b_n^\dagger$  ( $c_n^\dagger$ ) are interpreted as creation operators for fermions (anti-fermions) in the state determined by the wave number  $k_n$ , with corresponding annihilation operators  $b_n$  ( $c_n$ ).

Strictly speaking in obtaining the expressions above for  $H_D^0, Q, Q^5$  from the corresponding expressions in terms of the fields it is necessary to discard an infinite (c-number) sum, which is however divergent. Having done this, clearly the above expressions for  $H_D^0, Q, Q^5$  define operators which are well defined on domains containing  $\mathcal{F}$ . For example, on the unexcited states

$$\langle P | Q | P \rangle = 0, \quad \langle P | Q^5 | P \rangle = 2P$$

and, considering separately the case  $P \geq 0$  and  $P < 0$  we can verify that

$$\langle P | H_D^0 | P \rangle = \frac{2\pi}{L} P(P-1),$$

for all  $P \in \mathbb{Z}$ . (The state  $\Omega_1$  contains two particles of zero wave number and contributes nothing to the energy).

The Hilbert space completions of the various finite particle subspaces  $\mathcal{F}, \mathcal{F}_0$  and  $\mathcal{F}_0^P$  are written  $\mathcal{H} = \overline{\mathcal{F}}$  and

$$\begin{aligned} \mathcal{H}_0 &= \overline{\mathcal{F}_0} = \text{Ker } Q \subset \mathcal{H}, \\ \mathcal{H}_0^P &= \overline{\mathcal{F}_0^P} = \text{Ker } Q \cap \text{Ker } (Q^5 - 2P) \subset \mathcal{H}_0, \end{aligned} \quad (3.12)$$

where  $Q, Q^5$  are the (unbounded) self-adjoint operators on  $\mathcal{H}$  determined by the formal expressions in (3.10) on the domain  $\mathcal{F}$ . While  $\mathcal{H}_0$  and  $\{\mathcal{H}_0^P\}_{P \in \mathbb{Z}}$  are invariant Hilbert sub-spaces for the evolution  $e^{-itH_D^0}$  determined by the free Dirac Hamiltonian, it will transpire that only  $\mathcal{H}_0$  is invariant for the interacting theory even though classically the axial charge is conserved.

### 3.1 Gauge invariance and Schwinger regularization

As just mentioned it is conventional to discard infinite constants which arise in the definitions of  $Q, H_D^0$  in the second quantization scheme just outlined. Nevertheless, in treating the interaction with electromagnetic fields it is important to analyze carefully these infinite sums in order to assure gauge invariance. The first sign that care is needed comes from considering the action of the large gauge transformations  $g_N$  on the vacuum. The gauge transformation  $g_1$  acts classically as a phase rotation  $\psi \rightarrow g_1 \cdot \psi = \psi e^{2i\pi x/L}$  together with  $a \rightarrow g_1 \cdot a = a + 2\pi/L$ . In the second quantized formalism with an infinite Dirac sea this induces an shift operator:  $a_n^{L,R} \rightarrow a_{n-1}^{L,R}$ , with opposing effects on the kinetic energy (3.1) for the left and right components. In the positive energy representation a consideration of (3.5)-(3.6) above and the particle hole picture leads us to introduce a modified shift operator acting on the sequence of creation and annihilation operators as follows:

$$\begin{aligned} b_n &\rightarrow \Gamma b_n \Gamma^{-1} = b_{n-1}, \quad n \neq 0, & b_0 &\rightarrow \Gamma b_0 \Gamma^{-1} = c_1^\dagger \\ c_n &\rightarrow \Gamma c_n \Gamma^{-1} = c_{n+1}, \quad n \neq 0, & c_0 &\rightarrow \Gamma c_0 \Gamma^{-1} = b_{-1}^\dagger \end{aligned} \quad (3.13)$$

with corresponding relations for the adjoints:

$$\begin{aligned} b_n^\dagger &\rightarrow \Gamma b_n^\dagger \Gamma^{-1} = b_{n-1}^\dagger, \quad n \neq 0, & b_0^\dagger &\rightarrow \Gamma b_0^\dagger \Gamma^{-1} = c_1 \\ c_n^\dagger &\rightarrow \Gamma c_n^\dagger \Gamma^{-1} = c_{n+1}^\dagger, \quad n \neq 0, & c_0^\dagger &\rightarrow \Gamma c_0^\dagger \Gamma^{-1} = b_{-1} \end{aligned} \quad (3.14)$$

To these we append the action on the Dirac (bare) vacuum state:

$$\Gamma \Omega_0 = \Omega_{-1} = b_{-1}^\dagger c_1^\dagger \Omega_0. \quad (3.15)$$

Together these imply  $\Gamma \Omega_P = \Omega_{P-1}$  for all  $P$ . Similarly we obtain a corresponding modified shift action for  $\Gamma^{-1}$ :

$$\begin{aligned} b_n &\rightarrow \Gamma^{-1} b_n \Gamma = b_{n+1}, \quad n \neq -1, & b_{-1} &\rightarrow \Gamma^{-1} b_{-1} \Gamma = c_0^\dagger \\ c_n &\rightarrow \Gamma^{-1} c_n \Gamma = c_{n-1}, \quad n \neq 1, & c_1 &\rightarrow \Gamma^{-1} c_1 \Gamma = b_0^\dagger \end{aligned} \quad (3.16)$$

with corresponding relations for the adjoints:

$$\begin{aligned} b_n^\dagger &\rightarrow \Gamma^{-1} b_n^\dagger \Gamma = b_{n+1}^\dagger, \quad n \neq -1, & b_{-1}^\dagger &\rightarrow \Gamma^{-1} b_{-1}^\dagger \Gamma = c_0 \\ c_n^\dagger &\rightarrow \Gamma^{-1} c_n^\dagger \Gamma = c_{n-1}^\dagger, \quad n \neq 1, & c_1^\dagger &\rightarrow \Gamma^{-1} c_1^\dagger \Gamma = b_0 \end{aligned} \quad (3.17)$$

and the relation  $\Gamma^{-1} \cdot \Omega_0 = \Omega_1 = b_0^\dagger c_0^\dagger \Omega_0$  and  $\Gamma^{-1} \cdot \Omega_P = \Omega_{P+1}$  in general.

The gauge transformation acting on the creation annihilation operators and the vacuum according to (3.13)-(3.15) determines a unitary transformation on  $\mathcal{F}_0$  which extends to a unitary transformation  $\Gamma$  on  $\mathcal{H}_0$ . This transformation commutes with  $Q$  and so preserves  $\mathcal{H}_0$ , but it does not commute with  $Q^5$ : for example  $b_3^\dagger c_2^\dagger b_0^\dagger c_1^\dagger \Omega_0$  is mapped into  $b_2^\dagger c_3^\dagger c_2^\dagger b_{-1}^\dagger \Omega_0$ , with the chirality reducing by 2. Formally  $Q^5 \Gamma^{-1} = \Gamma^{-1} (Q^5 - 2)$  on  $\mathcal{F}_0$ .

The interpretation of all these formulae is that large gauge transformations can create and annihilate fermion/anti-fermion pairs in a way which seems naively to change the chiral charge: an anomaly. We show below that, nevertheless, the Schwinger regularization  $Q^{5,reg}$  of  $Q^5$  is unchanged by large gauge transformations.

As a first example of Schwinger regularization consider  $j^1$ : since  $\psi$  and  $\psi^\dagger$  are operator valued distributions the product  $\psi^\dagger(x) \gamma^5 \psi(x)$  cannot be taken without careful definition. On the other hand the tensor product  $\psi^\dagger(y) \gamma^5 \psi(x)$  has an unambiguous meaning as an operator valued distribution on the product space, and we can consider the limit  $y \rightarrow x$  by applying this to a sequence of test functions supported near the diagonal  $x = y$ . In order to assure gauge invariance it is necessary to insert Schwinger's line integral factor ([6]), leading finally to the tentative definition

$$j^{1,reg}(x) = \lim_{\theta \rightarrow 0} \int \psi^\dagger(y) e^{ia(y-x)} \gamma^5 \psi(x) \chi_\theta(x-y) dy \quad (3.18)$$

where  $\chi$  is a smooth periodic function which is zero outside of  $(-L/4, +L/4)$ , has  $\int_0^L \chi = 1$  and  $\chi_\theta(x) = \theta^{-1}\chi(x/\theta)$  so that the sequence  $\chi_\theta$  is an approximation to the identity as  $\theta \rightarrow 0$ . We now claim that the limit on the right hand side of (3.18) exists for all such  $\chi$  in the following sense: taking the matrix element between arbitrary vectors in  $\mathcal{F}$  the quantity

$$\frac{1}{L} \lim_{\theta \rightarrow 0} \iint \psi^\dagger(y) e^{ia(y-x)} \gamma^5 \psi(x) \chi_\theta(x-y) e^{-ik_m x} dy dx \quad (3.19)$$

is well defined  $\forall m \in \mathbb{Z}$  and independent of  $\chi$  with the properties above. We then define  $j^{1,reg}$  by the requirement that (3.19) equals its  $m^{th}$  Fourier coefficient  $j_1(m)$  - see (3.20). This seems to be sensible in the sense that (i) any acceptable definition of  $j^{1,reg}$  should satisfy this condition, and (ii) we will obtain an operator which does indeed satisfy this condition. The same method will then be used to define the Hamiltonian.

Fourier series give the momentum representation of the regularized axial charge density as

$$j^{1,reg}(x) = \sum j_1(m) e^{ik_m x}, \quad j_1(m) = \frac{1}{L} \int j^{1,reg}(x) e^{-ik_m x} dx. \quad (3.20)$$

Below we shall compute the matrix elements of the  $j_1(m)$  as defined by the Schwinger regularization (3.18). The most important is  $j_1(0)$ , or equivalently the regularized axial charge  $Q^{5,reg} = L j_1(0)$ , since this contains the anomalous contributions which are required to restore gauge invariance to the Hamiltonian (which was broken by the introduction of the vacuum in second quantization). After presenting these computations we then show that the other  $j_1(m)$  are given by their formal expressions without anomalous contributions.

### 3.2 Computation of regularized axial charge

We now discuss the axial charge itself which arises when  $m = 0$  in (3.19); explicitly it is

$$Q^{5,reg} = \frac{1}{L} \lim_{\theta \rightarrow 0} \sum_{n,n'} \iint (b_n^\dagger u_{n'}^\dagger e^{-ik_{n'} y} + c_{n'} v_n^\dagger e^{+ik_{n'} y}) e^{ia(y-x)} \gamma^5 \times \\ (b_n u_n e^{ik_n x} + c_n^\dagger v_n e^{-ik_n x}) \chi_\theta(x-y) dy dx, \quad (3.21)$$

with the understanding that the convergence is in the weak operator sense. In particular consider the matrix elements between two vectors  $\Omega_{\tilde{\mathbf{m}}, \tilde{\mathbf{n}}}$  and  $\Omega_{\tilde{\mathbf{m}}, \tilde{\mathbf{n}}}$  in  $\mathcal{F}$ . Recall that vectors  $\Omega_{\tilde{\mathbf{m}}, \tilde{\mathbf{n}}} \in \mathcal{F}$  are annihilated by all but a finite number of the  $b_n, c_{n'}$ : this ensures that expressions with annihilation operators on the right and creation operators on the left reduce to finite sums. In fact the only collection of terms inside  $\sum_{n,n'}$  which do not collapse to a finite sum for this reason are those involving  $c_n$  and  $c_n^\dagger$ , as all others are identically zero. For example the terms arising from the  $c_{n'}$  and  $b_n$  are

$$\lim_{\theta \rightarrow 0} \sum_{n,n'} \iint \langle \Omega_{\tilde{\mathbf{m}}, \tilde{\mathbf{n}}} | c_{n'} b_n | \Omega_{\tilde{\mathbf{m}}, \tilde{\mathbf{n}}} \rangle v_n^\dagger \gamma^5 u_n e^{ia(y-x)} e^{+ik_{n'} y} e^{ik_n x} \chi_\theta(x-y) dy dx$$

which is zero because  $v_n^\dagger \gamma^5 u_n = 0$  unless  $n, n'$  are either both positive or both negative in which case, writing  $z = y - x$  and changing variables, we have

$$\iint e^{ia(y-x)} e^{+ik_{n'} y} e^{ik_n x} \chi_\theta(x-y) dy dx = \iint e^{ia(z)} e^{+ik_{n'}(z)} \chi_\theta(z) e^{i(k_n + k_{n'})x} dz dx = 0.$$

Using  $\{c_{n'}, c_n^\dagger\} = \delta_{nn'}$  the collection of terms involving  $c_n$  and  $c_n^\dagger$  can be rearranged to have annihilation operators on the right at the expense of a c-number term  $C_A$ . To be precise we end up with

$$Q^{5,reg} = \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n + C_A, \quad (3.22)$$

where  $C_A = \frac{1}{L} \lim_{\theta \rightarrow 0} \sum_n \iint v_n^\dagger \gamma^5 v_n e^{+ik_n y} e^{ia(y-x)} e^{-ik_n x} \chi_\theta(x-y) dy dx.$

Introducing the periodic distributions defined by

$$\begin{aligned}\delta_+(z) &= \frac{1}{L} \sum_{n>0} e^{ik_n z} = \frac{1}{L} \lim_{\epsilon \downarrow 0} \frac{1}{e^{\frac{2\pi}{L}(\epsilon - iz)} - 1} = \frac{1}{L} \lim_{\epsilon \downarrow 0} \frac{1}{e^{\frac{-2i\pi}{L}(z + i\epsilon)} - 1} \\ \delta_-(z) &= \frac{1}{L} \sum_{n<0} e^{ik_n z} = \frac{1}{L} \lim_{\epsilon \downarrow 0} \frac{1}{e^{\frac{2\pi}{L}(\epsilon + iz)} - 1} = \frac{1}{L} \lim_{\epsilon \downarrow 0} \frac{1}{e^{\frac{2i\pi}{L}(z - i\epsilon)} - 1},\end{aligned}\tag{3.23}$$

so that  $\delta(z) = \frac{1}{L} + \delta_+(z) + \delta_-(z)$  is the  $L$ -periodic  $\delta$  function, we can write

$$C_A = \lim_{\theta \rightarrow 0} \iint (\delta_+(y-x) - \delta_-(y-x) - \frac{1}{L}) e^{ia(y-x)} \chi_\theta(x-y) dy dx.$$

The first  $\delta_\pm$  terms reduce to

$$\lim_{\theta \downarrow 0} \lim_{\epsilon \downarrow 0} \int \frac{1}{e^{\frac{2\pi}{L}(\epsilon \mp iz)} - 1} e^{iaz} \chi_\theta(z) dz = \lim_{\theta \downarrow 0} \lim_{\epsilon \downarrow 0} (I_{\epsilon, \theta}^\pm + II_{\epsilon, \theta}^\pm)$$

where

$$\begin{aligned}I_{\epsilon, \theta}^\pm &= \int \frac{e^{iaz} - 1}{e^{\frac{2\pi}{L}(\epsilon \mp iz)} - 1} \chi_\theta(z) dz \\ II_{\epsilon, \theta}^\pm &= \int \frac{1}{e^{\frac{2\pi}{L}(\epsilon \mp iz)} - 1} \chi_\theta(z) dz.\end{aligned}\tag{3.24}$$

For even  $\chi$  it is clear from parity considerations that  $II_{\epsilon, \theta}^+ - II_{\epsilon, \theta}^- = 0$ . For  $I_{\epsilon, \theta}^\pm$  notice that  $\frac{e^{iaz} - 1}{e^{\frac{2\pi}{L}(\epsilon \mp iz)} - 1} \rightarrow \frac{e^{iaz} - 1}{e^{\frac{\mp 2\pi iz}{L}} - 1}$  as  $\epsilon \downarrow 0$  for all non-zero  $z$  in the support of  $\chi_\theta$ . But also

$$\left| \frac{e^{iaz} - 1}{e^{\frac{2\pi}{L}(\epsilon \mp iz)} - 1} \right| \leq \frac{|e^{iaz} - 1|}{e^{2\pi\epsilon/L} |\sin \frac{2\pi z}{L}|} \leq \frac{c|z|}{|\sin \frac{2\pi z}{L}|} \leq c'$$

in the support of  $\chi_\theta$ , so using first the bounded convergence theorem and then the approximation to the identity theorem ([1, theorem 8.15]) we end up with:

$$\lim_{\theta \downarrow 0} \lim_{\epsilon \downarrow 0} I_{\epsilon, \theta}^\pm = \lim_{\theta \downarrow 0} \int \frac{e^{iaz} - 1}{e^{\frac{\mp 2\pi iz}{L}} - 1} \chi_\theta(z) dz = \mp \frac{aL}{2\pi}$$

leading finally to the following formula for the regularized axial charge operator:

$$Q^{5, reg} = \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n - \frac{aL}{\pi} - 1.\tag{3.25}$$

On the unexcited states we have

$$\langle P | Q^{5, reg} | P \rangle = 2P - \frac{aL}{\pi} - 1.\tag{3.26}$$

Notice the dependence on the background connection  $a$ , the independence of  $a$  of the naive expression for  $Q^5$  notwithstanding.

### 3.3 Computation of regularized kinetic energy

We consider the Schwinger regularization of the kinetic energy term  $H_D^0 = \int \psi^\dagger (-i\gamma^5 \partial) \psi$ , taking as starting point the definition analogous to (3.21):

$$\begin{aligned}H_D^{0, reg} &= \frac{1}{L} \lim_{\theta \rightarrow 0} \sum_{n, n'} \iint (b_{n'}^\dagger u_{n'}^\dagger e^{-ik_{n'} y} + c_{n'}^\dagger v_{n'}^\dagger e^{+ik_{n'} y}) e^{ia(y-x)} \gamma^5 \times \\ &\quad (b_n u_n k_n e^{ik_n x} - c_n^\dagger v_n k_n e^{-ik_n x}) \chi_\theta(x-y) dy dx,\end{aligned}\tag{3.27}$$

with the limit to be understood in the same sense as in (3.19). The structure is the same as (3.21) and by the same reasoning we have

$$H_D^{0,reg} = \sum_{n \geq 0} k_n b_n^\dagger b_n - \sum_{n < 0} k_n b_n^\dagger b_n + \sum_{n > 0} k_n c_n^\dagger c_n - \sum_{n \leq 0} k_n c_n^\dagger c_n + C'_A \quad (3.28)$$

Concentrating on the anomalous c-number term, we are led to consider

$$\begin{aligned} C'_A &= \lim_{\theta \rightarrow 0} \iint (\delta'_+(y-x) - \delta'_-(y-x)) i e^{ia(y-x)} \chi_\theta(y-x) dy dx \\ &= \lim_{\theta \rightarrow 0} \iint (\delta_+(y-x) - \delta_-(y-x)) (a e^{ia(y-x)} \chi_\theta(y-x) - i e^{ia(y-x)} \chi'_\theta(y-x)) dy dx. \end{aligned}$$

The contribution arising from the first term in the second bracket is  $-a^2 L/\pi$  by the previous calculation. For the contribution from the second term: first, we can subtract off an  $a$  independent c-number and replace  $e^{iaz}$  by  $e^{iaz} - 1$ , and then calculate the result to be

$$\begin{aligned} & \lim_{\theta \rightarrow 0} \lim_{\epsilon \rightarrow 0} -i \int \left( \frac{2i e^{\frac{2\pi\epsilon}{L}} \sin \frac{2\pi z}{L}}{e^{\frac{4\pi\epsilon}{L}} + 1 - 2e^{\frac{2\pi\epsilon}{L}} \cos \frac{2\pi z}{L}} \right) (e^{iaz} - 1) \chi'_\theta(z) dz \\ &= \lim_{\theta \rightarrow 0} \lim_{\epsilon \rightarrow 0} -i \int \left( \frac{2i e^{\frac{2\pi\epsilon}{L}} \sin \frac{2\pi z}{L}}{e^{\frac{4\pi\epsilon}{L}} + 1 - 2e^{\frac{2\pi\epsilon}{L}} \cos \frac{2\pi z}{L}} \right) (e^{iaz} - 1 - iaz) \chi'_\theta(z) dz \\ &= \lim_{\theta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int \left( \frac{\sin \frac{2\pi z}{L}}{1 - \cos \frac{2\pi z}{L}} \right) (e^{iaz} - 1 - iaz) \chi'_\theta(z) dz \\ &= - \int \frac{d}{dz} \left[ \left( \frac{\sin \frac{2\pi z}{L}}{1 - \cos \frac{2\pi z}{L}} \right) (e^{iaz} - 1 - iaz) \right] \chi_\theta(z) dz \end{aligned} \quad (3.29)$$

using, respectively, parity, bounded convergence theorem and integration by parts. The final limit can now be evaluated using again [1, theorem 8.15] to be  $a^2 L/(2\pi)$ . Adding this to the first term we end up with  $C'_A = -a^2 L/(2\pi)$  and so

$$\begin{aligned} H_D^{0,reg} &= \sum_{n \geq 0} k_n b_n^\dagger b_n - \sum_{n < 0} k_n b_n^\dagger b_n + \sum_{n > 0} k_n c_n^\dagger c_n - \sum_{n \leq 0} k_n c_n^\dagger c_n - \frac{a^2 L}{2\pi}, \\ &= \sum_{n \in \mathbb{Z}} |k_n| (b_n^\dagger b_n + c_n^\dagger c_n) - \frac{a^2 L}{2\pi}. \end{aligned} \quad (3.30)$$

The total Dirac Hamiltonian is  $H_D^a = H_D^0 - aQ^5$  so all together we posit the following formula for the regularized Hamiltonian:

$$H_D^{a,reg} = H_D^{0,reg} - aQ^{5,reg} \quad (3.31)$$

$$\begin{aligned} &= \sum_{n \geq 0} (k_n - a) b_n^\dagger b_n - \sum_{n < 0} (k_n - a) b_n^\dagger b_n + \sum_{n > 0} (k_n + a) c_n^\dagger c_n - \sum_{n \leq 0} (k_n + a) c_n^\dagger c_n + \frac{a^2 L}{2\pi} + a, \\ &= \sum_{n \in \mathbb{Z}} |k_n| (b_n^\dagger b_n + c_n^\dagger c_n) - \frac{a^2 L}{2\pi} - aQ^{5,reg}. \end{aligned} \quad (3.32)$$

On unexcited states:

$$\begin{aligned} \langle P | H_D^{a,reg} | P \rangle &= \frac{2\pi}{L} P(P-1) - \frac{a^2 L}{2\pi} - a \left( 2P - \frac{aL}{\pi} - 1 \right) = \frac{2\pi}{L} \left( P - \frac{aL}{2\pi} - \frac{1}{2} \right)^2 - \frac{\pi}{2L} \\ &= \frac{\pi}{2L} \left[ (\langle P | Q^{5,reg} | P \rangle)^2 - 1 \right]. \end{aligned} \quad (3.33)$$



### 3.4 Action of large gauge transformations

Under the gauge transformation  $g_1$  the regularized axial charge transforms to

$$\sum_{n \geq 0} b_n^\dagger b_n + c_1 c_1^\dagger - \sum_{n < -1} b_n^\dagger b_n - \sum_{n > 1} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n + b_{-1} b_{-1}^\dagger - \frac{(a + \frac{2\pi}{L})L}{\pi} - 1.$$

which equals  $Q^{5,reg}$  by the anti-commutation relations.

Similarly the regularized kinetic energy transforms to

$$\begin{aligned} & \sum_{n \geq 1} k_n b_{n-1}^\dagger b_{n-1} - \sum_{n < 0} k_n b_{n-1}^\dagger b_{n-1} + \sum_{n > 0} k_n c_{n+1}^\dagger c_{n+1} - \sum_{n < 0} k_n c_{n+1}^\dagger c_{n+1} - \frac{(a + \frac{2\pi}{L})^2 L}{2\pi} \\ &= \sum_{n \geq 0} (k_n + \frac{2\pi}{L}) b_n^\dagger b_n - \sum_{n < -1} (k_n + \frac{2\pi}{L}) b_n^\dagger b_n + \sum_{n > 1} (k_n - \frac{2\pi}{L}) c_n^\dagger c_n - \sum_{n \leq 0} (k_n - \frac{2\pi}{L}) c_n^\dagger c_n - \frac{(a + \frac{2\pi}{L})^2 L}{2\pi} \\ &= H_D^{0,reg} + k_{-1} b_{-1}^\dagger b_{-1} - k_1 c_1^\dagger c_1 + \frac{2\pi}{L} \left( \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < -1} b_n^\dagger b_n - \sum_{n > 1} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n \right) - 2a - \frac{2\pi}{L} \\ &= H_D^{0,reg} + \frac{2\pi}{L} \left( \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n \leq -1} b_n^\dagger b_n - \sum_{n \geq 1} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n \right) - 2a - \frac{2\pi}{L} \end{aligned} \quad (3.34)$$

Since  $aQ^{5,reg}$  transforms to  $(a + \frac{2\pi}{L})Q^{5,reg}$  it follows that  $H_D^{a,reg}$  transforms to  $H_D^{0,reg} - aQ^{5,reg} = H_D^{a,reg}$ , as required.

### 3.5 Computation of $j_1(m)$

We return to the regularized current  $j^{1,reg}(x) = \sum j_1(m) e^{ik_m x}$  from (3.20). We have already computed

$$j_1(0) = \frac{1}{L} Q^{5,reg} = \frac{1}{L} \left[ \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n - \frac{aL}{\pi} - 1 \right] \quad (3.35)$$

in section 3.2. The corresponding formulae for the other fourier modes are:

$$\begin{aligned} j_1(m) = & \frac{1}{L} \left[ \left( \sum_{n \geq 0 \text{ and } n' \geq 0} - \sum_{n < 0 \text{ and } n' < 0} \right) \delta_{n'+m-n,0} b_n^\dagger b_n \right. \\ & \left( \sum_{n > 0 \text{ and } n' \geq 0} - \sum_{n \leq 0 \text{ and } n' < 0} \right) \delta_{n'+m+n,0} b_n^\dagger c_n^\dagger \\ & \left( \sum_{n \geq 0 \text{ and } n' > 0} - \sum_{n < 0 \text{ and } n' \leq 0} \right) \delta_{n'-m+n,0} c_n b_n \\ & \left. \left( \sum_{n \leq 0 \text{ and } n' \leq 0} - \sum_{n > 0 \text{ and } n' > 0} \right) \delta_{n'-m-n,0} c_n^\dagger c_{n'} \right]. \end{aligned} \quad (3.36)$$

As already mentioned, there are no anomalous contributions to the  $j_1(m)$ , as defined by the Schwinger regularization, for  $m \neq 0$ . This is because the  $\{c_{n'}, c_n\} = \delta_{n'n}$  anti-commutator gives rise to a term involving the integrals  $\int e^{-ik_m x} \int v_n^\dagger \gamma^5 v_n e^{+ik_n y} e^{ia(y-x)} e^{-ik_n x} \chi_\theta(x-y) dy dx$  which are all identically zero for  $m \neq 0$ . Thus it remains to analyze the contributions from the terms involving the creation annihilation operators, and as before considering the matrix elements between two vectors  $\Omega_{\mathbf{m},\mathbf{n}}$  and  $\Omega_{\mathbf{m},\mathbf{n}}$  in  $\mathcal{F}$ , so that the vectors  $\Omega_{\mathbf{m},\mathbf{n}}$  are annihilated by all but a finite number of the  $b_n, c_{n'}$ .

We introduce similarly the regularized charge density by the formula analogous to (3.18) with momentum representation:

$$j^{0,reg}(x) = \sum j_0(m) e^{ik_m x}, \quad j_0(m) = \frac{1}{L} \int j^{0,reg}(x) e^{-ik_m x} dx. \quad (3.37)$$

Calculating as in section 3.2 leads to

$$j_0(0) = \frac{1}{L} Q^{reg} = \frac{1}{L} \left[ \sum_{n \geq 0} b_n^\dagger b_n + \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n - \sum_{n \leq 0} c_n^\dagger c_n \right], \quad (3.38)$$

(i.e. there is no anomaly in  $j^0$ ) and the corresponding formulae for the other fourier modes are:

$$\begin{aligned} j_0(m) = \frac{1}{L} \Big[ & \left( \sum_{n \geq 0 \text{ and } n' \geq 0} + \sum_{n < 0 \text{ and } n' < 0} \right) \delta_{n'+m-n,0} b_{n'}^\dagger b_n \\ & \left( \sum_{n > 0 \text{ and } n' \geq 0} + \sum_{n \leq 0 \text{ and } n' < 0} \right) \delta_{n'+m+n,0} b_{n'}^\dagger c_n^\dagger \\ & \left( \sum_{n \geq 0 \text{ and } n' > 0} + \sum_{n < 0 \text{ and } n' \leq 0} \right) \delta_{n'-m+n,0} c_{n'} b_n \\ & \left( - \sum_{n \leq 0 \text{ and } n' \leq 0} - \sum_{n > 0 \text{ and } n' > 0} \right) \delta_{n'-m-n,0} c_n^\dagger c_{n'} \Big]. \end{aligned} \quad (3.39)$$

Notice that all these expressions give rise to finite operator sums when matrix elements between elements of  $\mathcal{F}$  are computed - in fact the middle two lines define finite sums while the first and fourth, although in general unbounded, map the finite particle subspace to itself.

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